Finding Limits by Analytic Methods

Observing the graph of a function only, can be misleading at times when finding the limit of a function. It is possible to find limits using algebraic techniques and limit theorems.

You will learn to analyze limits by the following methods:

Methods to Analyze Limits:

1. Direct substitution
2. Properties of Limits
3. Algebraic simplification
4. Examining behavior from the left and right
5. Special trig limits (presented in Transcendental Functions lesson)
6. L'Hôpital's Rule (presented in Unit 3)

Substitution Theorem

If f is a polynomial function or rational function then $\lim_{x \to c} f(x) = f(c)$ provided that if f is a rational function the value of the denominator does not equal 0.

EX #1: Find each of the following limits analytically using direct substitution.

A. $\lim_{x \to 2}(3x^2 - 5x + 4) \approx 6$ $3(2)^2 - 5(2) + 4$ 6	B. $\lim_{x \to 2} \frac{x^3+1}{x+1} \approx 3$ $\frac{2^3+1}{2+1} = \frac{9}{3}$
C. $\lim_{x \to e} \frac{\ln x}{3x} \approx \frac{1}{3e}$ $\frac{\ln e}{3e} = \frac{1}{3e}$	D. $\lim_{x \to 4} \sqrt[3]{x+4} \approx 2$ $\sqrt[3]{4+4} \approx 2$
E. $\lim_{\theta \to \frac{\pi}{6}} \sin 2\theta$ $\frac{\sqrt{3}}{2}$ $\sin 2(\frac{\pi}{6}) = \sin(\frac{\pi}{3})$	F. $\lim_{x \to 5} \log_3(x+4) \approx 2$ $\log_3(5+4) = 2$

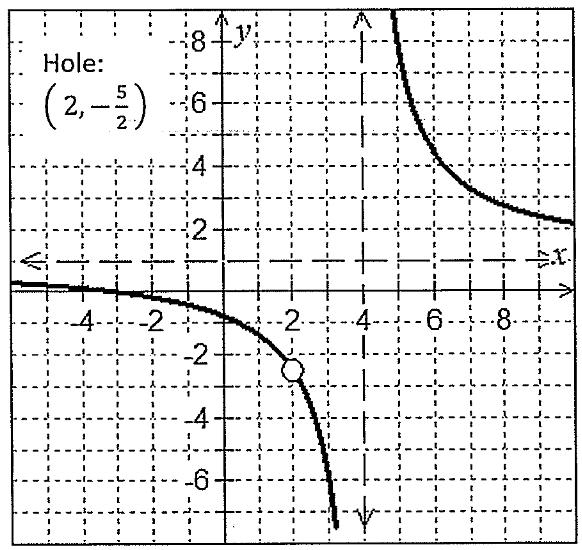
Finding Limits of Functions at Undefined Values

Consider the following cases and what happens when you try to evaluate limits by direct substitution.

EX #2: Algebraic Simplification: Factoring or Cancellation Technique

A. $\lim\limits_{x\to 2} \frac{x^2+x-6}{x^2-6x+8}$	B. $\lim\limits_{x\to 4^+} \frac{x^2+x-6}{x^2-6x+8}$	C. $\lim\limits_{x\to 4^-} \frac{x^2+x-6}{x^2-6x+8}$
$\frac{(x+3)(x-2)}{(x-4)(x-2)}$ Hole at $x=2$ To find limit We factor and simplify	$\frac{x+3(x-2)}{x-4(x-2)}$ Vertical Asymptote at $x=4$ To find limit we analyze right hand limit	Vertical asymptote at $x=4$ To find limit we analyze ~~right~~ left hand limit

Graphically, you can see the limits of the function shown at right. Just because a function is undefined at a value of x doesn't mean that you can't find the limit. Use the graph of the function to determine the value of each limit below.



$\lim\limits_{x\to 2} \frac{x^2+x-6}{x^2-6x+8} = \underline{-\frac{5}{2}}$

$\lim\limits_{x\to 4^+} \frac{x^2+x-6}{x^2-6x+8} = \underline{\infty}$

$\lim\limits_{x\to 4^-} \frac{x^2+x-6}{x^2-6x+8} = \underline{-\infty}$

What is the process for finding discontinuities of a rational function from pre-calculus?

You can perform the same algebraic analysis to find the limit of the removable, or point discontinuities and the non-removable, or infinite discontinuities using what we will call **Algebraic Simplifications**.

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 6x + 8}$$

1. Factor

$$\lim_{x \to 2} \frac{(x+3)(x-2)}{(x-4)(x-2)}$$

2. Simplify

$$\lim_{x \to 2} \frac{x+3}{x-4}$$

3. Substitute

$$\lim_{x \to 2} \frac{x+3}{x-4} = -\frac{5}{2}$$

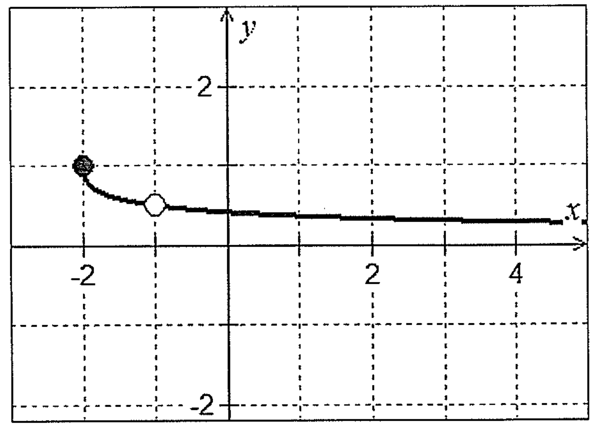
$$\frac{2+3}{2-4} = \frac{5}{-2}$$

Notice the simplified expression above and consider the behavior of this function. Graphically there is a non-removable discontinuity commonly called a Vertical asymptote at $x=4$. Because the y-values do not approach one specific value from both sides then the limit does not exist. By using the numerical, graphical, and algebraic techniques together, you can determine the behavior of the simplified function on either side of the vertical asymptote. This is true because the original function and the simplified function agree everywhere except at the point discontinuity at $x=2$.

Determining Behavior of a Function Using One-Sided Limits.			
$\lim_{x \to 4^+} \frac{x^2 + x - 6}{x^2 - 6x + 8}$		$\lim_{x \to 4^-} \frac{x^2 + x - 6}{x^2 - 6x + 8}$	
As $x \to 4^+$	Simplified function $\frac{x+3}{x-4}$	As $x \to 4^-$	Simplified function $\frac{x+3}{x-4}$
Pick a value: 4.1	$\frac{4.1+3}{4.1-4} \Rightarrow \frac{+}{+}$ As $x \to 4$, $y \to \infty$	Pick a value: 3.9	$\frac{3.9+3}{3.9-4} \Rightarrow \frac{+}{-}$ As $x \to 4$, $y \to -\infty$
$\lim_{x \to 4^+} f(x) = \infty$		$\lim_{x \to 4^-} f(x) = -\infty$	

EX #3: More Algebraic Simplification by Rationalization or Conjugate Method

A. The graph of $g(x) = \frac{\sqrt{x+2}-1}{x+1}$ is shown below. The technique of rationalization can be used to find the limit.



$$\lim_{x \to -1} \frac{\sqrt{x+2}-1}{x+1}$$

$\lim_{x \to -1} \frac{(\sqrt{x+2}-1)(\sqrt{x+2}+1)}{(x+1)(\sqrt{x+2}+1)}$

$\lim_{x \to -1} \frac{(x+2)-1}{(x+1)(\sqrt{x+2}+1)}$

$\lim_{x \to -1} \frac{x+1}{(x+1)(\sqrt{x+2}+1)}$

$\lim_{x \to -1} \frac{1}{\sqrt{x+2}+1} \approx \frac{1}{\sqrt{-1+2}+1} \approx \frac{1}{2}$

B. $\lim_{x \to 5} \frac{x-5}{3-\sqrt{x+4}}$

$\lim_{x \to 5} \frac{(x-5)(3+\sqrt{x+4})}{(3-\sqrt{x+4})(3+\sqrt{x+4}}$

$\lim_{x \to 5} \frac{(x-5)(3+\sqrt{x+4})}{9-(x+4)}$

$\lim_{x \to 5} \frac{(x-5)(3+\sqrt{x+4})}{5-x}$

$\lim_{x \to 5} \frac{(x-5)(3+\sqrt{x+4})}{-x+5}$

$\lim_{x \to 5} \frac{(x-5)(3+\sqrt{a+4})}{-(x-5)}$

$\lim_{x \to 5} -(3+\sqrt{x+4}$

$-(3+\sqrt{5+4})$

-6

EX #4: Find each of the following limits analytically. Show your algebraic steps.

A. $\lim_{x \to -2} x^3 + 3x^2 - 4x + 5$

$(-2)^3 + 3(-2)^2 - 4(-2) + 5$

B. $\lim_{x \to \frac{3}{2}} 2x^2(2x+3)$

$2(\frac{3}{2})^2(2(\frac{3}{2})+3)$

C. $\lim_{x\to 3}(5x+1)^{\frac{2}{3}}$

$\lim_{x\to 3}\sqrt[3]{(5\cdot 3+1)^2}$

$\sqrt[3]{(16)^2}$

D. $\lim_{x\to -4}\frac{2x^2+7x-4}{x^2-x-20}$

$\lim_{x\to -4}\frac{(2x-1)(x+4)}{(x-5)(x+4)}$

$\lim_{x\to -4}\frac{2x-1}{x-5}$

$\frac{(2\cdot -4)-1}{-4-5} = 1$

E. $\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}$

$\lim_{x\to 3}\frac{(\sqrt{x+1}-2)(\sqrt{x+1}+2)}{(x-3)(\sqrt{x+1}+2)}$

$\lim_{x\to 3}\frac{(x+1)-4}{(x-3)(\sqrt{x+1}+2)}$

$\lim_{x\to 3}\frac{x-3}{(x-3)\sqrt{x+1}+2}$

$\lim_{x\to 3}\frac{1}{\sqrt{x+1}+2}$

$\frac{1}{\sqrt{3+1}+2}$

$\frac{1}{4}$

F. $\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}$

$\lim_{x\to 0}\frac{(\sqrt{x+4}-2)(\sqrt{x+4}+2)}{x(\sqrt{x+4}+2)}$

$\lim_{x\to 0}\frac{(x+4)-4}{x(\sqrt{x+4}+2)}$

$\lim_{x\to 0}\frac{x}{x(\sqrt{x+4}+2}$

$\lim_{x\to 0}\frac{1}{\sqrt{x+4}+2}$

$\frac{1}{\sqrt{0+4}+2}$

$\frac{1}{4}$

G. $\lim_{x\to 0}\frac{\frac{1}{x}+\frac{1}{x+3}}{x}$

$\lim_{x\to 0}\frac{\frac{x+3}{x(x+3)}+\frac{x}{x(x+3)}}{x}$

$\lim_{x\to 0}\frac{\frac{2x+3}{x(x+3)}}{x}$

$\lim_{x\to 0}\frac{2x+3}{x^2(x+3)}$

Vertical Asymptote at $x=0$

$\lim_{x\to 0^-}f(x)=\infty$

$\lim_{x\to 0^+}f(x)=\infty$

$\lim_{x\to 0}f(x)=\infty$

H. $\lim_{x\to 2^+}\frac{3x^2-7x+2}{x^2-4}$

$\lim_{x\to 2^+}\frac{(3x-1)(x-2)}{(x+2)(x-2)}$

$\lim_{x\to 2^+}\frac{3x-1}{x+2}$

$\frac{3(2)-1}{2+2} = \frac{5}{4}$

J. $\lim_{x\to 5^+}\frac{4x+3}{x-5}$

Vertical Asymptote at $x=5$

Analyze

$\frac{4(5.1)+3}{5.1-5} \Rightarrow \frac{+}{+}$

As $x\to 5$, $y\to\infty$

$\lim_{x\to 5^+}\frac{4x+3}{x-5}=\infty$

K. $\lim_{x\to 5^-}\frac{4x+3}{x-5}$

Analyze

$\frac{4(4.9)+3}{4.9-5} \Rightarrow \frac{+}{-}$

As $x\to 5$, $y\to -\infty$

$\lim_{x\to 5^-}\frac{4x+3}{x-5}=-\infty$

EX #5: Finding limits analytically of piecewise functions. Show your algebraic steps.

A. Find $\lim\limits_{x\to 1} g(x)$ given,

$$g(x) = \begin{cases} x^3+1, & x>1 \quad \text{right} \\ x+1, & x\le 1 \quad \text{left} \end{cases}$$

$\lim\limits_{x\to 1^-} g(x) = 1+1 = 2$

$\lim\limits_{x\to 1^+} g(x) = 1^3+1 = 2$

$\lim\limits_{x\to 1} g(x) = 2$

B. Find $\lim\limits_{x\to 2} h(x)$ given,

$$h(x) = \begin{cases} x^2-4x+7, & x\le 2 \quad \text{left} \\ -x^2+4x-1, & x>2 \quad \text{right} \end{cases}$$

$\lim\limits_{x\to 2^-} h(x) = 2^2-4(2)+7 = 3$

$\lim\limits_{x\to 2^+} h(x) = -(2)^2+4(+2)-1 = 3$

$\lim\limits_{x\to 2} h(x) = 3$

C. Find $\lim\limits_{x\to -3} f(x)$ given,

$$f(x) = \begin{cases} \frac{x^2-9}{x+3}, & x<-3 \quad \text{left} \\ x^2+10x+15, & x\ge -3 \quad \text{right} \end{cases}$$

$\lim\limits_{x\to -3^-} f(x) = \frac{(x-3)(x+3)}{x+3} = -3-3 = -6$

$\lim\limits_{x\to -3^+} f(x) = (-3)^2+10(-3)+15 = -6$

$\lim\limits_{x\to 3} f(x) = -6$

D. Find $\lim\limits_{x\to 1} g(x)$ given,

$$g(x) = \begin{cases} x^2+4, & x\ne 1 \quad \text{right \& left} \\ 2, & x=1 \end{cases}$$

$\lim\limits_{x\to 1} g(x) = 1^2+4 = 5$

E. Find $\lim\limits_{x\to 0} f(x)$ given,

$$f(x) = \begin{cases} e^{2x}, & x<0 \quad \text{left} \\ xe^x, & x\ge 0 \quad \text{right} \end{cases}$$

$\lim\limits_{x\to 0^-} e^{2x} = e^0 = 1$

$\lim\limits_{x\to 0^+} xe^x = 0$

$\lim\limits_{x\to 0} f(x)$ DNE

F. Find $\lim\limits_{x\to 2} h(x)$ given,

$$h(x) = \begin{cases} 3-x, & x>2 \\ \frac{1}{2}x^2-1, & x\le 2 \end{cases}$$

$\lim\limits_{x\to 2^-} h(x) = \frac{1}{2}(2)^2-1 = 1$

$\lim\limits_{x\to 2^+} h(x) = 3-2 = 1$

$\lim\limits_{x\to 2} h(x) = 1$